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A class of exactly solved time-dependent quantum harmonic oscillators

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Abstract. We consider a class of time-dependent harmonic oscillators, $H(t) = p^2/2mt^\alpha + m\omega^2 t^b q^2/2$, whose mass and frequency vary as non-negative powers of time. Classically they describe damping oscillators slowly decaying as negative powers of time. Using the connection between classical and quantum harmonic oscillators we find analytically the Lewis–Riesenfeld invariants, obtain the exact quantum states, and compare these with the Caldirola–Kanai oscillator.

1. Introduction

The explicitly time-dependent quantum systems as non-stationary systems have been a long-standing mathematical problem not yet completely solved in general. The most general and frequently used method at present is the adiabatic method [1]. However, for explicitly time-dependent harmonic oscillators Lewis and Riesenfeld (LR) [2–3] have introduced an *important quantum mechanical invariant and found the exact quantum states in terms of* the invariant eigenstates up to some explicitly time-dependent phases. Since then numerous variants and applications [4–19] of the LR invariant method have been introduced and used. It still remains a difficult task to solve the nonlinear equation for the parameter entering the LR invariant equation. As one of the variants, we showed in the previous paper [20] that there exists a connection between classical and quantum harmonic oscillators based on the Lie algebra $so(2, 1)$. It is relatively easy to obtain exactly the integrals of the classical equation of motion for many physically interesting cases compared to the original nonlinear equation for the LR invariant.

There are also other methods to find the exact quantum states for time-dependent harmonic oscillators, one of which is to evaluate directly the evolution operator [21] using the Lie algebra $su(1, 1)$ that is isomorphic to $so(2, 1)$. Of particular practical use is the disentangled evolution operator which can be determined by the same integrals of the classical equation of motion [22]. Using the disentangled evolution operator one is able to find the exact quantum states in terms of the time-dependent number states of the new time-dependent number operator, which are the displaced and squeezed states of the initial states [23, 24]. It is also pointed out that the coherent states constructed from the number states of the LR invariant are the squeezed states [25].

In this paper, we shall find the LR invariant analytically for a class of time-dependent harmonic oscillators whose mass and frequency vary as non-negative powers of time. It is shown that these oscillators describe damping oscillators slowly decaying as negative powers of time classically. The method that will be used in this paper is the connection between classical and quantum harmonic oscillators that enable one to express the LR invariant in terms of the integrals of classical equations of motion [20]. Using the LR invariant we shall find the exact quantum states for these oscillators. Finally the connection is applied to the Caldirola–Kanai oscillator [26, 27], whose LR invariant and quantum states are well known.

2. Classical oscillators

We consider a class of explicitly time-dependent harmonic oscillators of the form

$$H(t) = \frac{p^2}{2mt^a} + \frac{m\omega^2 t^b q^2}{2} \tag{2.1}$$

whose mass and frequency vary as some non-negative powers of time. The Hamiltonian equations of motion yield

$$\ddot{q}(t) + \frac{a}{t}\dot{q}(t) + \omega^2 t^{b-a} q(t) = 0 \tag{2.2}$$

where the dots denote derivatives with respect to t. The solutions [28] are

$$q(t) = \begin{cases} t^\alpha J_\nu(z) \\ t^\alpha N_\nu(z) \end{cases} \tag{2.3}$$

where J_ν and N_ν are the Bessel functions, and

$$z = \beta t^\gamma \qquad \alpha = \frac{1-a}{2} \qquad \gamma = 1 + \frac{b-a}{2} \qquad \beta = \frac{\omega}{|\gamma|} \qquad \nu = \left|\frac{\alpha}{\gamma}\right| \tag{2.4}$$

unless $\gamma = 0$. For sufficiently large t, the solutions have the asymptotic forms $q(t) \approx t^{-b/2}$ for $\gamma > 0$ and $q(t) \approx t^{1-a}$ for $\gamma < 0$. Thus they describe damping oscillators slowly decaying as some negative powers of time compared with an exponentially decaying oscillator.

One can express the momentum and position in terms of initial data by

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} P_0(t,t_0) & P_1(t,t_0) \\ Q_1(t,t_0) & Q_0(t,t_0) \end{pmatrix} \begin{pmatrix} p(t_0) \\ q(t_0) \end{pmatrix} \tag{2.5}$$

where

$$\begin{aligned}
P_0(t,t_0) &= -\frac{\pi}{2\gamma} z^{-\nu}[(\alpha J_\nu(z) + \gamma z J'_\nu(z))N_\nu(z_0) - (\alpha N_\nu(z) + \gamma z N'_\nu(z))J_\nu(z_0)]z_0^\nu \\
P_1(t,t_0) &= \frac{\pi m \beta^{2\nu+1}}{2} z^{-\nu}[(\alpha J_\nu(z) + \gamma z J'_\nu(z))(\alpha N_\nu(z_0) \\
&\quad + \gamma z_0 N'_\nu(z_0)) - (\alpha N_\nu(z) + \gamma z N'_\nu(z))(\alpha J_\nu(z_0) + \gamma z_0 J'_\nu(z_0))]z_0^{-\nu} \\
Q_0(t,t_0) &= \frac{\pi}{2\gamma} z^\nu [J_\nu(z)(\alpha N_\nu(z_0) + \gamma z_0 N'_\nu(z_0)) - N_\nu(z)(\alpha J_\nu(z_0) + \gamma z_0 J'_\nu(z_0))]z_0^{-\nu} \\
Q_1(t,t_0) &= -\frac{\pi \beta^{-2\nu}}{2\gamma m} z^\nu [J_\nu(z)N_\nu(z_0) - N_\nu(z)J_\nu(z_0)]z_0^\nu
\end{aligned} \tag{2.6}$$

where primes denote derivatives with respect to z.

3. The LR invariant

The quantum harmonic oscillator corresponding to (2.1) obeys the Schrödinger equation (in units of $\hbar = 1$)

$$\mathrm{i}\frac{\partial}{\partial t}\psi(q,t) = \left[\frac{1}{mt^a}\frac{\hat{p}^2}{2} + m\omega^2 t^b \frac{\hat{q}^2}{2}\right]\psi(q,t) \tag{3.1}$$

where carets denote operators. It is well known that the quantum harmonic oscillator has a Lie algebra $so(2,1)$ with a basis [29–31]

$$L_1 = \frac{\hat{p}^2}{2} \qquad L_2 = \frac{\hat{p}\hat{q} + \hat{q}\hat{p}}{2} \qquad L_3 = \frac{\hat{q}^2}{2}. \tag{3.2}$$

Based on the Lie algebra, we find the LR invariant of the form

$$\hat{I}(t) = \sum_{k=1}^{3} g_k(t) L_k \tag{3.3}$$

that should satisfy the invariant equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{I}(t) = \frac{\partial}{\partial t}\hat{I}(t) - \mathrm{i}[\hat{I}(t), \hat{H}(t)] = 0 \tag{3.4}$$

which in component form reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix} = \begin{pmatrix} 2h_2(t) & -2h_1(t) & 0 \\ h_3(t) & 0 & -h_1(t) \\ 0 & 2h_3(t) & -2h_2(t) \end{pmatrix}\begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix}. \tag{3.5}$$

Using the connection between classical and quantum harmonic oscillators [20], one finds the LR invariant of the form

$$\begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix} = \begin{pmatrix} Q_0^2 & -2Q_1Q_0 & Q_1^2 \\ -P_1Q_0 & P_0Q_0 + P_1Q_1 & -P_0Q_1 \\ P_1^2 & -2P_1P_0 & P_0^2 \end{pmatrix}\begin{pmatrix} g_1(t_0) \\ g_2(t_0) \\ g_3(t_0) \end{pmatrix}. \tag{3.6}$$

Equation (3.6) prescribes the evolution of the LR invariant at an arbitrary time in terms of initial data. We may find exactly the LR invariant without the initial data by imposing a suitable boundary condition and taking a limiting procedure, which will be done below.

3.1. The case $\gamma > 0$

For sufficiently large t, that is, for large z, we have the asymptotic forms

$$g_1(t) \cong c_1\beta^{-2\nu+1}z^{2\nu-1} \qquad g_2(t) \cong c_2 \mathrm{O}\left(\frac{1}{z}\right) \qquad g_3(t) \cong c_3\beta^{2\nu-1}z^{-2\nu+1}. \tag{3.7}$$

Substituting the asymptotic forms [28]

$$\begin{aligned} J_\nu(z_0) &\cong \sqrt{\frac{2}{\pi z_0}}\cos\left(z_0 - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\ N_\nu(z_0) &\cong \sqrt{\frac{2}{\pi z_0}}\sin\left(z_0 - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \end{aligned} \tag{3.8}$$

for large z_0, choosing $c_3 = \beta^2\gamma^2 m^2 c_1$, and taking a limit $t_0, z_0 \to \infty$ in (3.6), we obtain finally the one-parameter-dependent LR invariant:

$$\begin{aligned} g_1(t) &= \frac{\pi\beta^{-2\nu+1}}{2} c_1 z^{2\nu}[J_\nu^2(z) + N_\nu^2(z)] \\ g_2(t) &= -\frac{\pi\beta\gamma m}{2} c_1[(\nu J_\nu(z) + zJ'_\nu(z))J_\nu(z) + (\nu N_\nu(z) + zN'_\nu(z))N_\nu(z)] \\ g_3(t) &= \frac{\pi\beta^{2\nu+1}\gamma^2 m^2}{2} c_1 z^{-2\nu}[(\nu J_\nu(z) + zJ'_\nu(z))^2 + (\nu N_\nu(z) + zN'_\nu(z))^2]. \end{aligned} \tag{3.9}$$

By directly substituting (3.9) into (3.5) and using the equation for the Bessel functions [28], one can show that (3.9) is indeed the LR invariant. The one-parameter-dependent LR invariant thus obtained is a consequence of the linear equation (3.5). We shall choose the constant $c_1 = 1/m$ for the sake of convenience, which makes the coefficient of $g_1(t)$ (or the most dominant term of the kinetic energy) of the LR invariant the same as that of the harmonic oscillator (3.1).

One also finds the LR invariant in a power series of z, again using the asymptotic forms for the Bessel functions [28]

$$\begin{aligned} J_\nu(z) &= \sqrt{\frac{2}{\pi z}}(W(\nu, z)\cos\chi - Z(\nu, z)\sin\chi) \\ N_\nu(z) &= \sqrt{\frac{2}{\pi z}}(W(\nu, z)\sin\chi + Z(\nu, z)\cos\chi) \end{aligned} \tag{3.10}$$

where $\chi = z - (\nu/2 + \frac{1}{4})\pi$, and

$$\begin{aligned} W(\nu, z) &= \sum_{k=0}^{\infty}(-1)^k \frac{(\nu, 2k)}{(2z)^{2k}} \\ Z(\nu, z) &= \sum_{k=0}^{\infty}(-1)^k \frac{(\nu, 2k+1)}{(2z)^{2k+1}} \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} (\nu, 0) &= 1 \\ (\nu, k) &= \frac{(4\nu^2 - 1)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2k-1)^2)}{k!2^{2k}}. \end{aligned} \tag{3.12}$$

Then, after some algebra, the LR invariant is given by

$$\begin{aligned} g_1(t) &= \frac{\beta^{-2\nu+1}}{m} z^{2\nu-1}[W^2(\nu, z) + Z^2(\nu, z)] \\ g_2(t) &= -\frac{\beta\gamma}{z}\left\{\left[\left(\nu - \frac{1}{2}\right)W(\nu, z) + zW'(\nu, z) - zZ(\nu, z)\right]W(\nu, z)\right. \\ &\quad \left. + \left[\left(\nu - \frac{1}{2}\right)Z(\nu, z) + zZ'(\nu, z) + zW(\nu, z)\right]Z(\nu, z)\right\} \\ g_3(t) &= \beta^{2\nu+1}\gamma^2 m z^{-2\nu-1}\left\{\left[\left(\nu - \frac{1}{2}\right)W(\nu, z) + zW'(\nu, z) - zZ(\nu, z)\right]^2\right. \\ &\quad \left. + \left[\left(\nu - \frac{1}{2}\right)Z(\nu, z) + zZ'(\nu, z) + zW(\nu, z)\right]^2\right\}. \end{aligned} \tag{3.13}$$

Note that the LR invariant will have a finite series expansion only when

$$\nu^2 = \left(\frac{2n-1}{2}\right)^2 \tag{3.14}$$

for some positive integer n.

3.2. The case of $\gamma < 0$

In this case $z \to 0$ as $t \to \infty$, so we use the asymptotic forms for the Bessel functions [28]

$$\begin{aligned} J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)} \\ N_\nu(z) &= -\pi\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}. \end{aligned} \tag{3.15}$$

For small z we have the asymptotic forms

$$g_1(t) \cong c_1\beta^{-4\nu}z^{4\nu} \qquad g_2(t) \cong c_2 O(z^{2\nu}) \qquad g_3(t) \cong c_3. \tag{3.16}$$

Then choosing $c_3 = (2\gamma m/\Gamma(\nu))^2(\beta/2)^\nu c_1$ and taking a limit $t_0 \to \infty$, $z_0 \to 0$, we obtain the following one-parameter-dependent LR invariant:

$$\begin{aligned} g_1(t) &= \beta^{-2\nu}c_1 z^{2\nu}J_\nu^2(z) \\ g_2(t) &= -\gamma m c_1 J_\nu(z)(\nu J_\nu(z) + zJ'_\nu(z)) \\ g_3(t) &= \beta^{2\nu}\gamma^2 m^2 c_1 z^{-2\nu}(\nu J_\nu(z) + zJ'_\nu(z))^2. \end{aligned} \tag{3.17}$$

In order to have the same coefficient for the most dominant terms of the kinetic energy of the Hamiltonian and LR invariant as in section 3.1, we choose the integration constant $c_1 = [\Gamma^2(\nu+1)/m](2/\beta)^{2\nu}$. Then we have the LR invariant of the form

$$\begin{aligned} g_1(t) &= \frac{2^{2\nu}\Gamma^2(\nu+1)}{m\beta^{4\nu}}z^{2\nu}J_\nu^2(z) \\ g_2(t) &= -\frac{2^{2\nu}\Gamma^2(\nu+1)\gamma}{\beta^{2\nu}}J_\nu(z)(\nu J_\nu(z) + zJ'_\nu(z)) \\ g_3(t) &= 2^{2\nu}\Gamma^2(\nu+1)\gamma^2 m z^{-2\nu}(\nu J_\nu(z) + zJ'_\nu(z))^2. \end{aligned} \tag{3.18}$$

A direct calculation shows that (3.18) satisfies (3.5).

4. Exact quantum states

It is well known that once the LR invariant is found, the exact quantum states of the Schrödinger equation in an analytic form follow immediately [2–3]. To construct the Fock space of the LR invariant, a generalized harmonic oscillator, we canonically transform the old operators into the new ones:

$$\hat{p}_{\text{new}} = \hat{p} + \frac{g_2(t)}{g_1(t)}\hat{q} \qquad \hat{q}_{\text{new}} = \hat{q}. \tag{4.1}$$

Using the fact that the eigenvalues of the LR invariant are constants of motion, which can also be shown directly, for example from (3.9), to have

$$\sqrt{(g_1(t)g_3(t) - g_2^2(t))} = \omega \tag{4.2}$$

we introduce the creation and annihilation operators by

$$\begin{aligned} \hat{p}_{\text{new}} &= \frac{\mathrm{i}}{\sqrt{2}}\sqrt{\frac{\omega}{g_1(t)}}(a^+(t) - a(t)) \\ \hat{q}_{\text{new}} &= \frac{1}{\sqrt{2}}\sqrt{\frac{g_1(t)}{\omega}}(a^+(t) + a(t)). \end{aligned} \tag{4.3}$$

Then the Fock space consists of the number states

$$\begin{aligned} \hat{I}(t)|n,t\rangle &= \omega(a^+(t)a(t) + \tfrac{1}{2})|n,t\rangle = \omega(n + \tfrac{1}{2})|n,t\rangle \\ a^+(t)|n,t\rangle &= \sqrt{n+1}|n+1,t\rangle \qquad a(t)|n,t\rangle = \sqrt{n}|n-1,t\rangle. \end{aligned} \tag{4.4}$$

It is obtained in [32] that

$$\begin{aligned} \frac{\partial}{\partial t}a^+(t) &= \epsilon(t)a^+(t) + \delta(t)a(t) \\ \frac{\partial}{\partial t}a(t) &= \delta^*(t)a^+(t) + \epsilon^*(t)a(t) \end{aligned} \tag{4.5}$$

and

$$\left\langle n,t\left|\frac{\partial}{\partial t}\right|n,t\right\rangle = \epsilon(t)(n + \tfrac{1}{2}) \tag{4.6}$$

where

$$\begin{aligned} \epsilon(t) &= -\frac{\mathrm{i}g_1(t)}{2\omega}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g_2(t)}{g_1(t)}\right) \\ \delta(t) &= -\frac{\mathrm{i}g_1(t)}{2\omega}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g_2(t)}{g_1(t)}\right) - \frac{1}{2g_1(t)}\frac{\mathrm{d}g_1(t)}{\mathrm{d}t}. \end{aligned} \tag{4.7}$$

The expectation values of the Hamiltonian of harmonic oscillator (3.1) are given by

$$\begin{aligned} \langle n,t|\hat{H}(t)|n,t\rangle &= \frac{(\omega^2 + g_2^2(t))h_1(t) + g_1^2(t)h_3(t)}{2\omega g_1(t)}\left(n + \frac{1}{2}\right) \\ &= h(t)(n + \tfrac{1}{2}) \end{aligned} \tag{4.8}$$

where

$$h_1(t) = \frac{1}{mt^a} \qquad h_3(t) = m\omega^2 t^b. \tag{4.9}$$

Then it is known that the exact quantum states of the Schrödinger equation (3.1) are

$$\psi_n(q,t) = \exp\left(-\mathrm{i}\int(h(t) - \mathrm{i}\epsilon(t))(n + \tfrac{1}{2})\right)|n,t\rangle \tag{4.10}$$

and in coordinate representation

$$\begin{aligned} \psi_n(q,t) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\sqrt{\frac{\omega}{g_1(t)}}\exp\left[-\mathrm{i}\int(h(t) - \mathrm{i}\epsilon(t))\left(n + \frac{1}{2}\right)\right] \\ \times \exp\left(-\frac{\omega}{2g_1(t)}q^2\right)H_n\left(\sqrt{\frac{\omega}{g_1(t)}}q\right). \end{aligned} \tag{4.11}$$

5. Comparison with the Caldirola–Kanai oscillator

The Caldirola–Kanai oscillator [27, 28] which has an exponentially increasing mass and frequency has been studied through various methods [33–38] and is one of the most typical time-dependent quantum systems whose exact quantum states and LR invariant are known.

The Caldirola–Kanai oscillator

$$H(t) = \frac{p^2}{2me^{2at}} + \frac{m\omega^2 e^{2at} q^2}{2} \tag{5.1}$$

whose equation of motion is

$$\ddot{q}(t) + a\dot{q}(t) + \omega^2 q(t) = 0 \tag{5.2}$$

describes a damping harmonic oscillator. The solutions are

$$q(t) = \begin{cases} e^{-at}\cos\Omega t \\ e^{-at}\sin\Omega t \end{cases} \tag{5.3}$$

where

$$\Omega = \sqrt{\omega^2 - a^2}. \tag{5.4}$$

Here we shall consider the case of weak damping ($\omega > a$) only, but the result below can be easily extended to the other case. The momentum and position are determined by (2.5) as

$$\begin{aligned}
P_0(t, t_0) &= \frac{1}{\Omega} e^{a(t-t_0)} [\Omega\cos\Omega(t-t_0) - a\sin\Omega(t-t_0)] \\
P_1(t, t_0) &= -\frac{m\omega^2}{\Omega} e^{a(t+t_0)} \sin\Omega(t-t_0) \\
Q_0(t, t_0) &= \frac{1}{\Omega} e^{-a(t-t_0)} [\Omega\cos\Omega(t-t_0) + a\sin\Omega(t-t_0)] \\
Q_1(t, t_0) &= \frac{1}{m\Omega} e^{-a(t+t_0)} \sin\Omega(t-t_0).
\end{aligned} \tag{5.5}$$

For sufficiently large t, we have the asymptotic forms for the LR invariant

$$g_1(t) \cong c_1 e^{-2at} \qquad g_2(t) \cong c_2 \qquad g_3(t) \cong c_3 e^{2at}. \tag{5.6}$$

Setting $c_2 = mac_1$ and $c_3 = m^2\omega^2 c_1$, we obtain the one-parameter LR invariant

$$g_1(t) = c_1 e^{-2at} \qquad g_2(t) = mac_1 \qquad g_3(t) = m^2\omega^2 c_1 e^{2at}. \tag{5.7}$$

Choosing the constant $c_1 = 1/m$, the LR invariant then has the form

$$g_1(t) = \frac{1}{me^{2at}} \qquad g_2(t) = a \qquad g_3(t) = m\omega^2 e^{2at}. \tag{5.8}$$

Note that the LR invariant can now be rewritten as [33]

$$\hat{I}(t) = \hat{H}(t) + a\frac{\hat{p}\hat{q} + \hat{q}\hat{p}}{2}. \tag{5.9}$$

We find the frequency of the LR invariant as

$$\sqrt{g_1(t)g_3(t) - g_2^2(t)} = \Omega \tag{5.10}$$

and

$$\epsilon(t) = -\frac{a^2}{\Omega}\mathrm{i} \qquad h(t) = \frac{\omega^2}{\Omega}. \tag{5.11}$$

Therefore, from (4.10) we obtain the exact eigenfunctions for the Caldirola–Kanai oscillator

$$\psi_n(q,t) = \sqrt{\frac{m\Omega}{2^n n!\sqrt{\pi}}} \exp\left\{\left[at - \mathrm{i}\Omega\left(n + \frac{1}{2}\right)t - \frac{m\Omega \mathrm{e}^{2at}}{2}q^2\right]\right\} H_n\left(\sqrt{m\Omega}\mathrm{e}^{at}q\right) \tag{5.12}$$

where H_n are the Hermite polynomials. Note that the exact eigenfunctions differ from those in [33] by some trivial factors.

6. Conclusion

In this paper we found the LR invariant analytically for a class of time-dependent harmonic oscillators (2.1) which have time-dependent mass and frequency as some non-negative powers of time. Depending on the power laws of time we obtained the one-parameter-dependent LR invariants (3.9), (3.17) and (A8). With a choice of the parameter that makes the kinetic energy term of the LR invariant approach that of the Hamiltonian for sufficiently large times, the LR invariant took the forms (3.13) and (3.18). In particular, the LR invariant has a finite series expansion if the condition (3.14) is satisfied. In the asymptotic region of sufficiently large times it is observed that the most dominant terms of the LR invariant when expanded asymptotically constitute the Hamiltonian itself, and that the asymptotic region is the adiabatic region for the superadiabatic expansion [39]. It is also found that the power law time-dependent harmonic oscillators have all the features of the Caldirola–Kanai oscillator as non-stationary dissipative quantum systems except that now the power law dependence has replaced the exponential dependence.

One may construct the coherent states for the time-dependent oscillator (3.1) out of the number states (4.10) of the LR invariant according to [8]. It is pointed out that these coherent states at a late time are simply the displaced and squeezed states of an early state [25]. This can be understood from the fact that the creation and annihilation operators (4.3) of the LR invariant depend explicitly on time so that the vacuum at the early time evolves into the squeezed vacuum at the late time [40]. One may also evaluate the disentangled evolution operator according to [24] using the integrals of the classical equation of motion (2.3), express the evolution operator as the product of the squeezing and Weyl displacement operators, and show that the exact quantum states are the displaced and squeezed states. This evolution operator method differs from the LR invariant method used in this paper in many respects.

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Appendix. The case $\gamma = 0$

The case $\gamma = 0$ occurs when $a = b + 2$. The equation of motion now becomes

$$\ddot{q}(t) + \frac{a}{t}\dot{q}(t) + \frac{\omega^2}{t^2}q(t) = 0. \tag{A1}$$

There are power law solutions

$$q(t) = \begin{cases} t^{\alpha_-} \\ t^{\alpha_+} \end{cases} \tag{A2}$$

where

$$\alpha_\pm = \frac{-(a-1) \pm \sqrt{[(a-1)^2 - 4\omega^2]}}{2} \tag{A3}$$

unless $a = 1 \pm 2\omega$. We shall restrict ourselves below to the special case of $a > 1 + 2\omega$. The momentum and position can be expressed as

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} P_0(t, t_0) & P_1(t, t_0) \\ Q_1(t, t_0) & Q_0(t, t_0) \end{pmatrix} \begin{pmatrix} p(t_0) \\ q(t_0) \end{pmatrix} \tag{A4}$$

where

$$\begin{aligned} P_0(t, t_0) &= \frac{1}{\Delta\alpha}(\alpha_+ t^{\alpha_+} t_0^{\alpha_-} - \alpha_- t^{\alpha_-} t_0^{\alpha_+}) \\ P_1(t, t_0) &= -\frac{m\omega^2}{\Delta\alpha} t^{a-1}(t^{\alpha_+} t_0^{\alpha_-} - t^{\alpha_-} t_0^{\alpha_+}) t_0^{a-1} \\ Q_0(t, t_0) &= -\frac{1}{\Delta\alpha}(\alpha_- t^{\alpha_+} t_0^{\alpha_-} - \alpha_+ t^{\alpha_-} t_0^{\alpha_+}) t_0^{a-1} \\ Q_1(t, t_0) &= \frac{1}{m\Delta\alpha}(t^{\alpha_+} t_0^{\alpha_-} - t^{\alpha_-} t_0^{\alpha_+}) \end{aligned} \tag{A5}$$

where

$$\Delta\alpha = \alpha_+ - \alpha_-. \tag{A6}$$

After some trial and error we are able to find the asymptotic forms

$$g_1(t_0) \cong c_1 t_0^{2\alpha_-} \qquad g_2(t_0) \cong c_2 t_0^{-\Delta\alpha} \qquad g_3(t_0) = c_3 t_0^{-2\alpha_+}. \tag{A7}$$

Substituting (A7) into (3.6), taking a limit $t_0 \to \infty$, and choosing $c_2 = (m\alpha_+/2)c_1$, $c_3 = m^2\alpha_+^2 c_1$, we find the one-parameter-dependent LR invariant without the initial data:

$$\begin{aligned} g_1(t) &= 3\left(\frac{\alpha_+}{\Delta\alpha}\right)^2 c_1 t^{2\alpha_-} \\ g_2(t) &= -3\frac{m\omega^2\alpha_+}{(\Delta\alpha)^2} c_1 t^{-\Delta\alpha} \\ g_3(t) &= 3\left(\frac{m\omega^2}{\Delta\alpha}\right)^2 c_1 t^{-2\alpha_+}. \end{aligned} \tag{A8}$$

By directly putting (A8) into (3.5) we show that (A8) indeed satisfies the LR invariant equation.

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